The Sparse-Grid Combination Technique Applied to Time-Dependent Advection Problems

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Abstract. In the numerical technique considered in this paper, time-stepping is performed on a set of semi-coarsened space grids. At given time levels the solutions on the different space grids are combined to obtain the asymptotic convergence of a single, fine uniform grid. We present error estimates for the two-dimensional, spatially constant-coefficient model problem and discuss numerical examples. A spatially variable-coefficient problem (Molenkamp-Crowley test) is used to assess the practical merits of the technique. The combination technique is shown to be more efficient than the single-grid approach, yet for the Molenkamp-Crowley test standard Richardson extrapolation is still more efficient than the combination technique. However, parallelization is expected to significantly improve the combination technique's performance.
Keywords: advection problems, sparse grids, combination techniques, error analysis.

1 Introduction

Sparse grids were introduced by Zenger [1] in 1990 to reduce the number of degrees of freedom in finite-element calculations. The combination technique, as introduced in 1992 by Griebel, Schneider and Zenger [2], can be seen as a practical implementation of the sparse-grid technique. In the combination technique, the final solution is a linear combination of solutions on semi-coarsened grids, where the coefficients of the combination are chosen such that there is a canceling in leading-order error terms.

In literature the combination technique has already been analyzed for elliptic problems. In [3], promising numerical results are presented for the combination technique applied to a constant-coefficient advection equation. The current work differs from [3] in that it also presents error estimates and, although we do not present error estimates for spatially variable-coefficients, we do study this case numerically with the Molenkamp-Crowley test. Furthermore, a time-dependent coefficient case is considered. For the derivation of the error estimates we refer to [4]. In the current work and in [4] we neglect the representation error that is due to the combination technique, this error is analyzed in [5].

2 Combination technique

In the two-dimensional combination technique approximate solutions $\omega^{l,m}$ on semi-coarsened grids $\Omega^{l,m}$ are combined according to

$$\widehat{\omega}^{N,N} = \sum_{l+m=N} P^{N,N}\omega^{l,m} - \sum_{l+m=N-1} P^{N,N}\omega^{l,m}, \tag{1}$$

to obtain a single, more accurate solution $\widehat{\omega}^{N,N}$. Here upper indices label the mesh widths of $\Omega^{l,m}$ according to $h_x = 2^{-l}H$ and $h_y = 2^{-m}H$, where H is the mesh width of the uniform root grid $\Omega^{0,0}$. We denote the mesh width of the finest grid $\Omega^{N,N}$ by $h = 2^{-N}H$. The $P^{N,N}$ are prolongation operators that map grid functions from $\Omega^{l,m}$ onto $\Omega^{N,N}$.

In the current work the $\omega^{l,m}$ are semi-discrete (continuous in time) approximate solutions to an initial-value problem that we integrate from time $t = 0$ up to $t = 1$. In the numerical implementation time integration is done numerically, but the corresponding temporal discretization error is negligible with respect to the spatial discretization error. Straightforward application of the combination technique implies that the $\omega^{l,m}$ are integrated up to $t = 1$ and then combined according to (1). We also consider application with $M - 1$ *intermediate combinations* at equidistant points in time. The solutions $\omega^{l,m}$ are then integrated up to $t = 1/M$. The resulting solution is restricted onto the grids $\Omega^{l,m}$, yielding new initial solutions $\omega^{l,m}$, which are further integrated up to $t = 2/M$, etcetera. This process is repeated until a combined solution $\widehat{\omega}^{N,N}$ at time level $t = 1$ is obtained.

3 Error estimates

The focus lies on the pure initial-value problem for the spatially-constant coefficient, 2D advection equation

$$c_t + ac_x + bc_y = 0. \tag{2}$$

Equation (2) is integrated in time with the third-order upwind-biased discretization on the spatial domain $[-1, 1] \times [-1, 1]$. This yields the semi-discrete, approximate solution $\omega(t)$. For a single-grid technique the corresponding (global) discretization error $d(t) \equiv \omega(t) - c_h(t)$ is, in leading order, given by

$$d(t) = -\frac{th^3}{12}\left(|a|\,\partial_x^4 + |b|\,\partial_y^4\right) c_h(t) + \mathcal{O}(h^4), \tag{3}$$

provided a and b are independent of time.

The combined discretization error $\widehat{d}(t)$, i.e., the error due to the combination technique, with M combinations, is in leading order given by

$$\begin{aligned} \widehat{d}(t) &= -\tfrac{th^3}{12}(|a|\,\partial_x^4 + |b|\,\partial_y^4)c_h(t) \\ &\quad + \tfrac{t^2h^3}{144}\tfrac{H^3}{M}\,|ab|\,(1 - 7\log_2 \tfrac{H}{h})\partial_x^4\partial_y^4 c_h(t) + \mathcal{O}(h^4 \log_2 \tfrac{H}{h}), \end{aligned} \tag{4}$$

provided that the order of the interpolation used for prolongation is greater than the order of the discretization. For the derivations of above expressions we refer to [4], where we also give an error analysis that is valid when a and b are time-dependent.

4 Numerical results

All the numerical results presented in this paper were obtained with fourth-order accurate explicit Runge-Kutta time integration with time step $\Delta t = 0.1\min(h_x, h_y)$ which satisfies the CFL condition for all considered test cases. The time discretization error is always negligible compared to the spatial discretization error. For spatial discretization we have used third-order upwind discretization. The prolongations are done with fourth-order interpolation.

4.1 Test cases

We consider the following four test cases :

1. Horizontal advection, characterized by $a = 1/2$, $b = 0$.
2. Diagonal advection with $a = b = 1/2$.
3. Time-dependent advection with $(a, b) = \begin{cases} (0,2), & 0 \le t < 1/4. \\ (2,0), & 1/4 \le t < 1/2. \\ (0,-2), & 1/2 \le t < 3/4. \\ (-2,0), & 3/4 \le t < 1. \end{cases}$
4. The Molenkamp-Crowley test case with $a = 2\pi y$, $b = -2\pi x$.

The Gaussian initial profile for test cases 1-3 is depicted in Fig. 1(a) and the Gaussian initial profile for test case 4 is depicted in Fig. 1(d). All test cases are integrated up to $t = 1$ and have $-1 \le x, y \le 1$.

4.2 Results

For the horizontal test case the single-grid (SG) error and the errors due to the combination technique with (ICT) and without (CT) intermediate combinations are all practically equal and are perfectly described by the analytical prediction (4). The combination technique does not introduce any additional error relative to the single-grid technique because the second term in (4) vanishes due to $b = 0$. The combination technique works very well for this fully grid-aligned test case, as can be seen in Fig. 2(a). Fig. 2(a) also shows that 7 intermediate combinations do not improve the efficiency for the horizontal test case. In fact, the ICT results coincide with the CT results.

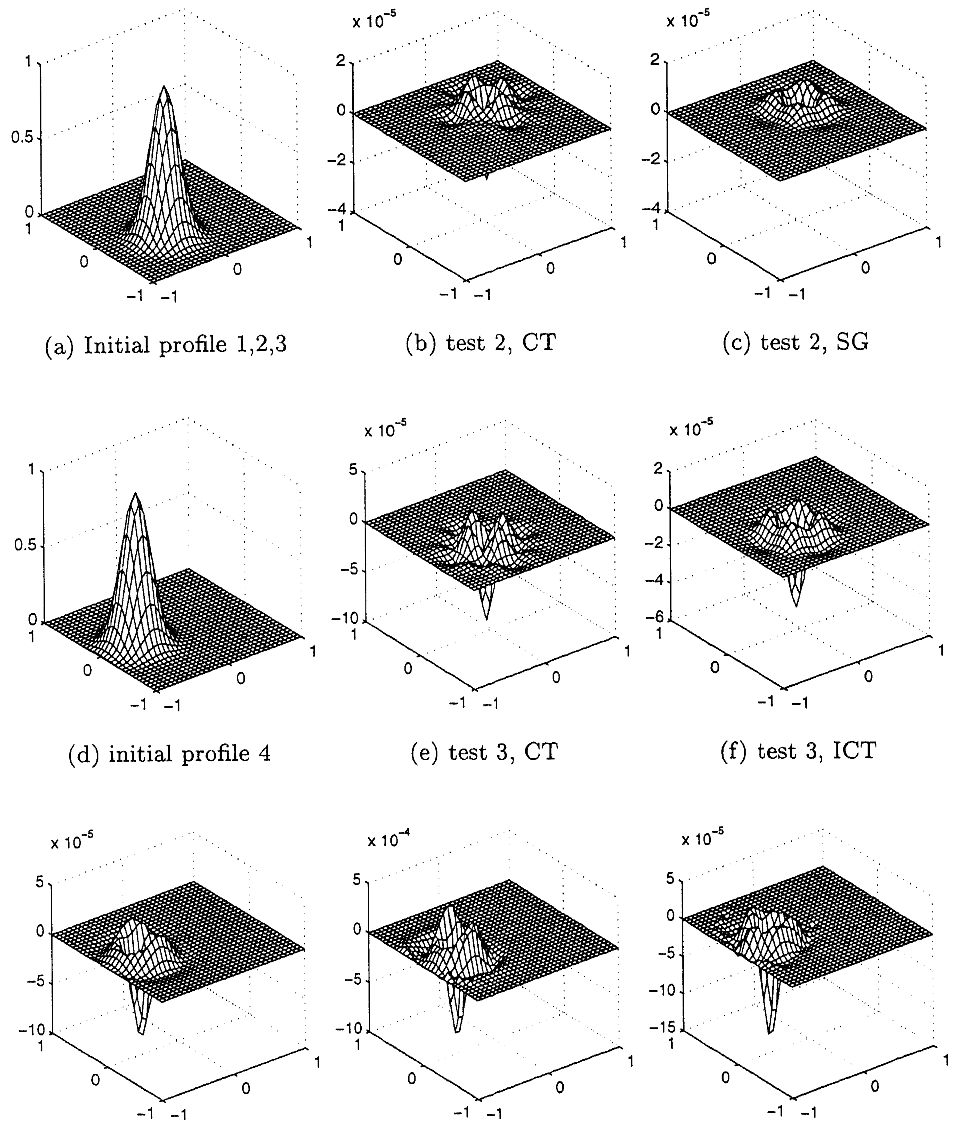
For the diagonal test case, error profiles are shown for the CT and the SG technique in Figs. 1(b) and 1(c), respectively. We see that for this test case the CT error is somewhat larger than the SG error and has a different shape. Fig. 2(b) shows that the combination technique can be made more accurate by applying 7 intermediate combinations. Fig. 3(a) shows that the ICT error converges to the single-grid error as the number of combinations is increased. The first couple of combinations strongly decrease the error, a further increase in the number of combinations does not decrease the error much further.

For the time-dependent test case the error profiles for the CT and the ICT are plotted in Figs. 1(e) and 1(f), respectively. We see that making intermediate combinations influences both the shape and size of the error. Note that Figs. 2(b) and 2(c) are similar, i.e., just like the diagonal test case the time-dependent test case is solved more efficiently with intermediate combinations (ICT) than without (CT). However, the reason for the efficiency of the ICT is somewhat more complex for the time-dependent test case than for the diagonal test case. As we can see from Fig. 3(b) the ICT error does not decrease monotonically with the number of combinations and this is correctly predicted by our theory. We can see that when a multiple of four combinations is made, the ICT error becomes equal to the single-grid error. The time-dependent test case is then effectively split into two horizontal and two vertical advection problems and these are solved very well by the combination technique, as we know from the first test case.

Error profiles for the Molenkamp-Crowley test case are shown in Figs. 1(g), 1(h) and 1(i) for the SG, CT and ICT, respectively. We see that the CT error is larger than the SG error, but intermediate combinations help considerably, i.e., the ICT error lies much closer to the SG error than to the CT error. Fig. 2(d) shows that the Molenkamp-Crowley test case is a tough case to solve efficiently with the combination technique, i.e., the CT is less efficient than the SG technique, but the ICT is more efficient than the SG technique. Fig. 3(c) shows how the ICT error decreases with increasing number of combinations.

In [6] Rüde points out that simple Richardson extrapolation is in fact more efficient than the combination technique for the solution of a smooth Poisson problem. To see how Richardson extrapolation would perform for the Molenkamp-Crowley test case, we considered the following Richardson extrapolant $\omega_R^{N,N} \equiv \frac{8}{7}\omega^{N,N} - \frac{1}{7}P^{N,N}\omega^{N-1,N-1}$; it cancels the leading third-order term in (3).

Fig. 2(d) clearly shows that Richardson extrapolation is very efficient for the Molenkamp-Crowley test case, much more so than the combination technique, even though we expect the combination technique to be superior to Richardson extrapolation in the asymptotic limit $h \to 0$. For the Molenkamp-Crowley test case, without parallelization and on grids of practically relevant mesh width, the combination technique can not compete with classical Richardson extrapolation.



(a) Initial profile 1,2,3 (b) test 2, CT (c) test 2, SG

(d) initial profile 4 (e) test 3, CT (f) test 3, ICT

(g) test 4, SG (h) test 4, CT (i) test 4, ICT

Fig. 1. Initial profiles and numerically observed errors for the single-grid technique (SG), the combination technique (CT) and the combination technique with 7 intermediate combinations (ICT) applied to the test cases.

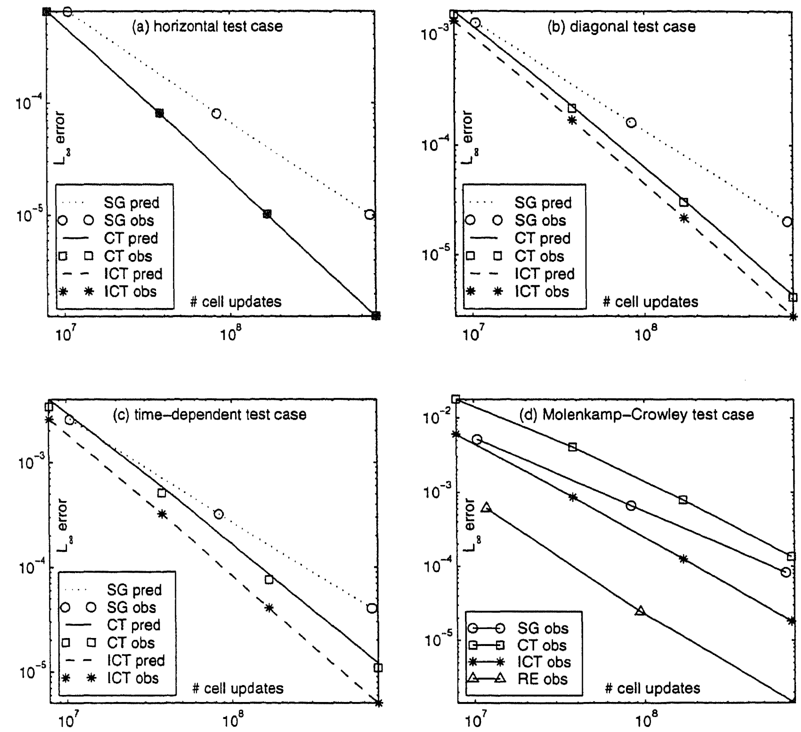


Fig. 2. Numerically observed (obs) and analytically predicted (pred) performance of the single-grid technique (SG), the combination technique (CT), the combination technique with 7 intermediate combinations (ICT) and the Richardson extrapolation technique (RE) applied to the test cases 1-4.

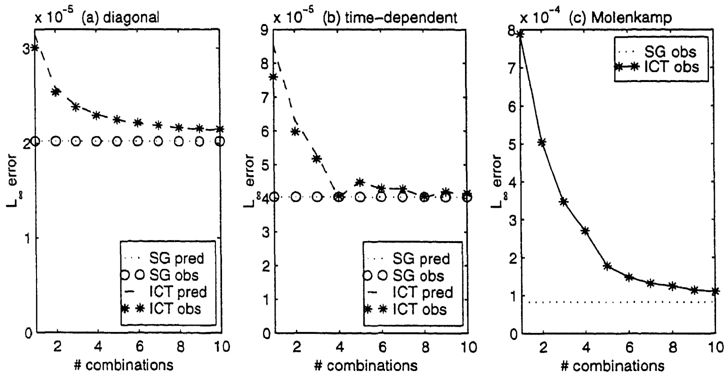


Fig. 3. L_∞ error versus number of combinations for the test cases 2-3.

5 Conclusions

We have presented leading-order expressions for the error that is introduced when a spatially constant-coefficient advection equation is solved with the combination technique. We have accounted for intermediate combinations. (In [4] we also present error expressions that are valid for time-dependent coefficients.)

For the spatially-independent test cases, the derived error expressions perfectly predict the outcome of the numerical tests. For these test cases the combination technique outperforms the single-grid technique even without intermediate combinations, especially the grid-aligned advection test is solved very well by the combination technique. Intermediate combinations are required to outperform the single-grid technique on the Molenkamp-Crowley test. For this test simple Richardson extrapolation proved more efficient than the combination technique, even though the combination technique is expected to be more efficient in the asymptotic limit $h \to 0$. Rüde made the same observation for a smooth Poisson problem in [6].

When going to three spatial dimensions (or even higher dimensional problems), the combination technique will perform significantly better. Furthermore, very significant gains in performance can be obtained when the combination technique is parallelized.

References

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